

BRAIDED MULTIPLICATIVE UNITARIES AS REGULAR OBJECTS

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ABSTRACT. We use the theory of regular objects in tensor categories to clarify the passage between braided multiplicative unitaries and multiplicative unitaries with projection. The braided multiplicative unitary and its semidirect product multiplicative unitary have the same Hilbert space representations. We also show that the multiplicative unitaries associated to two regular objects for the same tensor category are equivalent and hence generate isomorphic C^* -quantum groups. In particular, a C^* -quantum group is determined uniquely by its tensor category of representations on Hilbert space, and any functor between representation categories that does not change the underlying Hilbert spaces comes from a morphism of C^* -quantum groups.

1. INTRODUCTION

The Tannaka–Krein Theorem by Woronowicz [10] recovers a compact quantum group from its tensor category of finite-dimensional representations, together with the forgetful functor to the tensor category of Hilbert spaces. We shall prove an analogue of this result for C^* -quantum groups, that is, quantum groups generated by manageable multiplicative unitaries. Our result asserts that an isomorphism between the tensor categories of Hilbert space representations that does not change the underlying Hilbert spaces lifts to an isomorphism of the underlying Hopf $*$ -algebras. More generally, we shall explain how to extract multiplicative unitaries from representation categories and how to lift tensor functors between representation categories to morphisms of multiplicative unitaries.

This article grew out of a suggestion by David Bücher to clarify the construction of a semidirect product multiplicative unitary from a braided multiplicative unitary in [5, 8]. A braided multiplicative unitary is supposed to describe a braided C^* -quantum group, which should be a Yetter–Drinfeld algebra over some other C^* -quantum group, equipped with a comultiplication $B \rightarrow B \boxtimes B$ into its Yetter–Drinfeld twisted tensor square. The semidirect product is constructed in [5, 8] by writing down a unitary and checking that it is multiplicative. The data of a braided multiplicative unitary consists of four unitaries, subject to seven conditions. All four unitaries must appear in the explicit formula, and all seven conditions must be used in the proof that the semidirect product is multiplicative. Thus the direct verification in [5] is rather complicated. Here we offer a conceptual explanation for this construction.

The main idea behind this is the theory of regular objects in tensor categories by Pinzari and Roberts [7]. We prefer to call them natural right absorbers because the adjective “regular” is already used for too many other purposes. A natural right

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absorber in \mathfrak{C} gives rise to a multiplicative unitary \mathbb{W} and a tensor functor from \mathfrak{C} to the tensor category of Hilbert space representations of \mathbb{W} . Representations of the semidirect product multiplicative unitary should be equivalent to representations of the braided multiplicative unitary. This idea already appears in a special case in [1]. Here we extend this result to the general case. Starting with a braided multiplicative unitary, we define its representation category and describe a natural right absorber in it by combining two rather obvious pieces. The corresponding multiplicative unitary turns out to be the semidirect product. We also show that the functor from representations of the braided multiplicative unitary to representations of the semidirect product is an isomorphism of categories. The most difficult point here is to prove that any representation of the semidirect product comes from a representation of the braided multiplicative unitary.

The semidirect product comes with a projection, which is another multiplicative unitary linked to it by pentagon-like equations. We interpret this projection through a projection on the representation category. More generally, we show that any tensor functor between representation categories that does not change the underlying Hilbert spaces lifts to a morphism between the associated multiplicative unitaries as defined in [2, 6]. This also implies the weak Tannaka–Krein Theorem for C^* -quantum groups mentioned above. And it gives yet another equivalent description of quantum groups with projection.

2. NATURAL RIGHT ABSORBERS IN HILBERT SPACE TENSOR CATEGORIES

We are going to recall the notion of a (right) regular object of a tensor category from [7]. We call such an object a natural right absorber, avoiding the overused adjective “regular”. Going beyond [7], we show that different natural right absorbers give isomorphic multiplicative unitaries with respect to the morphisms of C^* -quantum groups defined in [2, 6]. We also add a further equivalent description of such quantum group morphisms through functors between representation categories, and we show that isomorphic multiplicative unitaries generate isomorphic C^* -quantum groups as expected.

Notation 2.1. Let \mathfrak{Hilb} denote the W^* -category of Hilbert spaces. This is a symmetric monoidal category for the usual tensor product \otimes of Hilbert spaces, with the obvious associator $(\mathcal{H}_1 \otimes \mathcal{H}_2) \otimes \mathcal{H}_3 \cong \mathcal{H}_1 \otimes (\mathcal{H}_2 \otimes \mathcal{H}_3)$, the obvious unit transformations $\mathbb{C} \otimes \mathcal{H} \cong \mathcal{H} \cong \mathcal{H} \otimes \mathbb{C}$, and the obvious symmetric braiding

$$\Sigma: \mathcal{H}_1 \otimes \mathcal{H}_2 \rightarrow \mathcal{H}_2 \otimes \mathcal{H}_1, \quad x_1 \otimes x_2 \mapsto x_2 \otimes x_1.$$

Let \mathfrak{C} be a W^* -category with a faithful forgetful functor $\mathfrak{For}: \mathfrak{C} \rightarrow \mathfrak{Hilb}$. Faithfulness allows us to assume that $\mathfrak{C}(x_1, x_2) \subseteq \mathbb{B}(\mathfrak{For}(x_1), \mathfrak{For}(x_2))$ for all objects $x_1, x_2 \in \mathfrak{C}$ (we write \in for objects of categories, \in for arrows). We say that $a \in \mathbb{B}(\mathfrak{For}(x_1), \mathfrak{For}(x_2))$ *comes from* \mathfrak{C} if it belongs to the image of $\mathfrak{C}(x_1, x_2)$. We think of objects in \mathfrak{C} as Hilbert spaces with some extra structure, such as a representation of a (braided) multiplicative unitary; the morphisms are those bounded linear maps that preserve this extra structure. Motivated by this interpretation, we assume the following throughout this article:

Assumption 2.2. *If $\mathfrak{For}(x) = \mathfrak{For}(x')$ and the identity map on this Hilbert space comes from an arrow $x \rightarrow x'$, then $x = x'$.*

We also want a functor $\tau: \mathfrak{Hilb} \rightarrow \mathfrak{C}$ with $\mathfrak{For} \circ \tau = \text{id}_{\mathfrak{Hilb}}$. Thus τ acts as the identity on arrows, and the arrows $\tau(\mathcal{H}_1) \rightarrow \tau(\mathcal{H}_2)$ in \mathfrak{C} are exactly all bounded linear operators $\mathcal{H}_1 \rightarrow \mathcal{H}_2$. We abbreviate $\tau x := \tau \circ \mathfrak{For}(x)$ for $x \in \mathfrak{C}$. We interpret τ as the functor that equips a Hilbert space \mathcal{H} with the “trivial” extra structure to get an object in \mathfrak{C} .

We assume that \mathfrak{C} is also a monoidal category, but not necessarily braided, such that both \mathfrak{For} and τ are strict monoidal functors. This means, first, that $\mathfrak{For}(x_1 \otimes x_2) = \mathfrak{For}(x_1) \otimes \mathfrak{For}(x_2)$ for all $x_1, x_2 \in \mathfrak{C}$ and $\tau(\mathcal{H}_1 \otimes \mathcal{H}_2) = \tau(\mathcal{H}_1) \otimes \tau(\mathcal{H}_2)$ for all $\mathcal{H}_1, \mathcal{H}_2 \in \mathfrak{Hilb}$. Secondly, that the tensor unit in \mathfrak{C} is $\tau\mathbb{C}$, which \mathfrak{For} maps back to the tensor unit in \mathfrak{Hilb} . Thirdly, \mathfrak{For} and τ map associators and unit transformations in \mathfrak{C} to the obvious associators and unit transformations in \mathfrak{Hilb} . Finally, we require the following assumption, which is trivial to check in all cases we shall consider:

Assumption 2.3. *Let $x_1, x_2, y \in \mathfrak{C}$ and let $a: \mathfrak{For}(x_1) \rightarrow \mathfrak{For}(x_2)$ be such that $a \otimes \text{id}$ comes from an arrow $x_1 \otimes y \rightarrow x_2 \otimes y$ in \mathfrak{C} or $\text{id} \otimes a$ comes from an arrow $y \otimes x_1 \rightarrow y \otimes x_2$ in \mathfrak{C} . Then a itself comes from an arrow $x_1 \rightarrow x_2$ in \mathfrak{C} .*

Definition 2.4. A *Hilbert space tensor category* is a monoidal W^* -category \mathfrak{C} with a faithful, strict monoidal functor $\mathfrak{For}: \mathfrak{C} \rightarrow \mathfrak{Hilb}$ and a strict monoidal functor $\tau: \mathfrak{Hilb} \rightarrow \mathfrak{C}$ satisfying $\mathfrak{For} \circ \tau = \text{id}_{\mathfrak{Hilb}}$ and Assumptions 2.2 and 2.3.

Example 2.5. Let $\mathbb{W} \in \mathcal{U}(\mathcal{H} \otimes \mathcal{H})$ be a multiplicative unitary. Let $\mathfrak{Rep}(\mathbb{W})$ be the W^* -category of its (right) Hilbert space representations, with intertwiners as arrows. That is, the objects are pairs $(\mathcal{K}, \mathbb{U})$ where \mathcal{K} is a Hilbert space and $\mathbb{U} \in \mathcal{U}(\mathcal{K} \otimes \mathcal{H})$ satisfies $\mathbb{W}_{23}\mathbb{U}_{12} = \mathbb{U}_{12}\mathbb{U}_{13}\mathbb{W}_{23}$ in $\mathcal{U}(\mathcal{K} \otimes \mathcal{H} \otimes \mathcal{H})$. The arrows $(\mathcal{K}^1, \mathbb{U}^1) \rightarrow (\mathcal{K}^2, \mathbb{U}^2)$ are operators $a \in \mathbb{B}(\mathcal{K}^1, \mathcal{K}^2)$ with $\mathbb{U}^2 a_1 = a_1 \mathbb{U}^1$, where $a_1 := a \otimes \text{id}_{\mathcal{H}}$ in the leg numbering notation. The forgetful functor $\mathfrak{Rep}(\mathbb{W}) \rightarrow \mathfrak{Hilb}$ forgets the representation, and $\tau(\mathcal{K}) := (\mathcal{K}, 1)$. The tensor product of two representations $\mathbb{U}^i \in \mathcal{U}(\mathcal{K}^i \otimes \mathcal{H})$, $i = 1, 2$, is $\mathbb{U}^1 \oplus \mathbb{U}^2 := \mathbb{U}_{13}^1 \mathbb{U}_{23}^2 \in \mathcal{U}(\mathcal{K}^1 \otimes \mathcal{K}^2 \otimes \mathcal{H})$. Quick computations show that this is again a representation, that \oplus is associative, and that $\tau(\mathbb{C})$ is a tensor unit, with the usual associator and unit transformations from \mathfrak{Hilb} . Since an operator of the form $a_1 \in \mathbb{B}(\mathcal{K}^1 \otimes \mathcal{K}^2)$ for $a \in \mathbb{B}(\mathcal{K}^1)$ commutes with \mathbb{U}_{23}^2 , it is an intertwiner for $\mathbb{U}_{13}^1 \mathbb{U}_{23}^2$ if and only if a is one for \mathbb{U}^1 . Hence Assumption 2.3 holds. Assumption 2.2 holds because our objects are indeed Hilbert spaces with extra structure.

Lemma 2.6. *Let $x_1, x_2 \in \mathfrak{C}$, $\mathcal{H} \in \mathfrak{Hilb}$. Then an operator $a: \mathfrak{For}(x_1) \otimes \mathcal{H} \rightarrow \mathfrak{For}(x_2)$ comes from an arrow $\hat{a} \in \mathfrak{C}(x_1 \otimes \tau\mathcal{H}, x_2)$ if and only if the operators $a_\eta: \mathfrak{For}(x_1) \rightarrow \mathfrak{For}(x_2)$, $\xi \mapsto a(\xi \otimes \eta)$, come from arrows in $\mathfrak{C}(x_1, x_2)$ for all $\eta \in \mathcal{H}$. Analogous statements hold for operators $\mathcal{H} \otimes \mathfrak{For}(x_1) \rightarrow \mathfrak{For}(x_2)$, $\mathfrak{For}(x_1) \rightarrow \mathfrak{For}(x_2) \otimes \mathcal{H}$, $\mathfrak{For}(x_1) \rightarrow \mathcal{H} \otimes \mathfrak{For}(x_2)$.*

Proof. If there is an arrow $\hat{a} \in \mathfrak{C}(x_1 \otimes \tau\mathcal{H}, x_2)$, then we get arrows \hat{a}_η in $\mathfrak{C}(x_1, x_2)$ with $\mathfrak{For}(\hat{a}_\eta) = a_\eta$ by taking $\hat{a}_\eta := \hat{a} \circ (\text{id}_{x_1} \otimes \tau(|\eta\rangle))$, where $|\eta\rangle: \mathbb{C} \rightarrow \mathcal{H}$, $\lambda \mapsto \lambda\eta$, and where we implicitly identify $x_1 \cong x_1 \otimes \tau\mathbb{C}$.

For the converse, let $(P_i)_{i \in I}$ be a net of finite-rank projections converging weakly to the identity operator on \mathcal{H} . Then we get endomorphisms $\text{id}_{x_1} \otimes \tau(P_i) \in \mathfrak{C}(x_1 \otimes \tau\mathcal{H}, x_2)$, which still converge weakly to the identity operator. Since $\mathfrak{C}(x_1 \otimes \tau\mathcal{H}, x_2)$ is weakly closed, it suffices to lift $a \circ (\text{id}_{x_1} \otimes \tau(P_i))$ to an arrow in \mathfrak{C} for all i and then take a limit. Writing the finite-rank projection P_i as a sum of rank-1 projections, we further reduce to the lifting of an operator of the form $a \circ |\eta\rangle\langle\eta|$ for a unit vector $\eta \in \mathcal{H}$. This is lifted by the following composite in \mathfrak{C} :

$$x_1 \otimes \tau(\mathcal{H}) \xrightarrow{\text{id} \otimes \tau(|\eta\rangle)} x_1 \otimes \tau(\mathbb{C}) \cong x_1 \xrightarrow{\hat{a}_\eta} x_2. \quad \square$$

Remark 2.7. The functor τ is unique if it exists. Let \mathcal{H} be a Hilbert space. Then any bounded linear operator $\mathbb{C} \rightarrow \mathcal{H}$ comes from an arrow $\tau(\mathbb{C}) \rightarrow \tau(\mathcal{H})$ in \mathfrak{C} . Conversely, let x be an object of \mathfrak{C} with $\mathfrak{For}(x) = \mathcal{H}$ such that any bounded linear map $\mathbb{C} \rightarrow \mathcal{H}$ comes from an arrow $\mathbb{C} \rightarrow x$. Hence the identity map $\tau(\mathcal{H}) \rightarrow x$ comes from an arrow in \mathfrak{C} by Lemma 2.6. Then $\tau(\mathcal{H}) = x$ by Assumption 2.2.

Given objects $x_1, x_2, x_3, y_1, y_2, y_3 \in \mathfrak{C}$, there are canonical maps

$$\begin{aligned} \mathfrak{C}(x_1 \otimes x_2, y_1 \otimes y_2) &\rightarrow \mathfrak{C}(x_1 \otimes x_2 \otimes x_3, y_1 \otimes y_2 \otimes x_3), & T &\mapsto T_{12} = T \otimes \text{id}_{x_3}, \\ \mathfrak{C}(x_2 \otimes x_3, y_2 \otimes y_3) &\rightarrow \mathfrak{C}(x_1 \otimes x_2 \otimes x_3, x_1 \otimes y_2 \otimes y_3), & T &\mapsto T_{23} = \text{id}_{x_1} \otimes T. \end{aligned}$$

An arrow T_{13} , however, cannot always be defined because this would require a braiding on \mathfrak{C} . Nevertheless, the operator T_{13} may be defined if the object in the middle is of the form $\tau\mathcal{H}$. Lemma 2.6 implies that the flip operator

$$\Sigma: \mathfrak{For}(x) \otimes \mathcal{H} \rightarrow \mathcal{H} \otimes \mathfrak{For}(x), \quad \xi \otimes \eta \mapsto \eta \otimes \xi,$$

comes from an arrow in $\mathfrak{C}(x \otimes \tau\mathcal{H}, \tau\mathcal{H} \otimes x)$ for all $x \in \mathfrak{C}$, $\mathcal{H} \in \mathfrak{Hilb}$. We use these arrows in \mathfrak{C} to define

$$\begin{aligned} \mathfrak{C}(x_1 \otimes x_2, y_1 \otimes y_2) &\rightarrow \mathfrak{C}(x_1 \otimes \tau\mathcal{H} \otimes x_2, y_1 \otimes \tau\mathcal{H} \otimes y_2), \\ T &\mapsto T_{13} := \Sigma_{23} T_{12} \Sigma_{23} = \Sigma_{12} T_{23} \Sigma_{12}. \end{aligned}$$

Definition 2.8. Let \mathfrak{C} be a Hilbert space tensor category as above. A *natural right absorber* in \mathfrak{C} is an object $\rho \in \mathfrak{C}$ together with unitaries

$$U^x: x \otimes \rho \rightarrow \tau(x) \otimes \rho \quad \text{for all } x \in \mathfrak{C}$$

with the following properties:

- (2.8.1) the unitaries U^x are natural in the sense that the following diagram commutes for any arrow $a \in \mathfrak{C}(x_1, x_2)$, $x_1, x_2 \in \mathfrak{C}$:

$$\begin{array}{ccc} x_1 \otimes \rho & \xrightarrow[\cong]{U^{x_1}} & \tau(x_1) \otimes \rho \\ a \otimes \text{id}_\rho \downarrow & & \downarrow a \otimes \text{id}_\rho = \tau(a) \otimes \text{id}_\rho \\ x_2 \otimes \rho & \xrightarrow[\cong]{U^{x_2}} & \tau(x_2) \otimes \rho \end{array}$$

- (2.8.2) for $x_1, x_2 \in \mathfrak{C}$ the unitaries $U^{x_1 \otimes x_2}$, U^{x_1} and U^{x_2} are related by the following commuting diagram of unitaries:

$$\begin{array}{ccc} x_1 \otimes x_2 \otimes \rho & \xrightarrow{U^{x_1 \otimes x_2}} & \tau(x_1 \otimes x_2) \otimes \rho \\ U_{23}^{x_2} \downarrow & & \parallel \\ x_1 \otimes \tau(x_2) \otimes \rho & \xrightarrow{U_{13}^{x_1}} & \tau(x_1) \otimes \tau(x_2) \otimes \rho \end{array}$$

Lemma 2.9. If ρ and $(U^x)_{x \in \mathfrak{C}}$ are a natural right absorber for \mathfrak{C} , then $U^{\tau(\mathcal{H})} = \text{id}_{\tau(\mathcal{H}) \otimes \rho}$ for any Hilbert space \mathcal{H} .

Proof. Assumption (2.8.2) for $x_1 = x_2 = \mathbb{C} = \tau(\mathbb{C})$ implies $U^{\tau(\mathbb{C})} = \text{id}_{\mathbb{C}}$. Any vector $\xi \in \mathcal{H}$ gives an arrow $|\xi\rangle: \tau(\mathbb{C}) \rightarrow \tau(\mathcal{H})$. The naturality assumption (2.8.1) applied to these arrows gives $U^{\tau(\mathcal{H})}(\xi \otimes \eta) = \xi \otimes \eta$ for all $\xi \in \mathcal{H}$, $\eta \in \mathfrak{For}(\rho)$. Thus $U^{\tau(\mathcal{H})} = \text{id}$. \square

Example 2.10. Let \mathbb{W} be a multiplicative unitary and let $\mathfrak{C} = \mathfrak{Rep}(\mathbb{W})$ as in Example 2.5. The pentagon equation says that the unitary \mathbb{W} is also a representation of itself. A unitary $\mathbb{U} \in \mathcal{U}(\mathcal{K} \otimes \mathcal{H})$ is a representation if and only if it is an intertwiner

$$(\mathcal{K} \otimes \mathcal{H}, \mathbb{U}_{13} \mathbb{W}_{23}) = (\mathcal{K} \otimes \mathcal{H}, \mathbb{U} \oplus \mathbb{W}) \rightarrow (\mathcal{K} \otimes \mathcal{H}, \text{id}_{\mathcal{K}} \oplus \mathbb{W}) = (\mathcal{K} \otimes \mathcal{H}, \mathbb{W}_{23}).$$

We claim that \mathbb{W} with the family of arrows $\mathbb{U}: (\mathcal{K}, \mathbb{U}) \otimes (\mathcal{H}, \mathbb{W}) \rightarrow (\mathcal{K}, \text{id}_{\mathcal{K}}) \otimes (\mathcal{H}, \mathbb{W})$ is a natural right absorber in $\mathfrak{Rep}(\mathbb{W})$. First, the arrows in $\mathfrak{Rep}(\mathbb{W})$ are exactly those operators for which the arrows \mathbb{U} above are natural. Secondly, the tensor product of two representations is defined exactly so as to verify (2.8.2).

Proposition 2.11 ([7, Theorem 2.1]). *Let $(\mathfrak{C}, \mathfrak{Fot}, \tau, \otimes)$ be a Hilbert space tensor category and let ρ and $(U^x)_{x \in \mathfrak{C}}$ be a natural right absorber for \mathfrak{C} . For $x \in \mathfrak{C}$, let $\mathcal{H}^x := \mathfrak{Fot}(x)$, and let us also write U^x for $\mathfrak{Fot}(U^x) \in \mathcal{U}(\mathcal{H}^x \otimes \mathcal{H}^\rho)$. Then U^ρ is a multiplicative unitary, and U^x for $x \in \mathfrak{C}$ is a right representation of U^ρ . This construction gives a fully faithful, strict tensor functor from \mathfrak{C} to the tensor category $\mathfrak{Rep}(U^\rho)$ of representations of the multiplicative unitary U^ρ , which intertwines the forgetful functors from \mathfrak{C} and $\mathfrak{Rep}(U^\rho)$ to \mathfrak{Hilb} .*

Proof. The condition (2.8.2) and Lemma 2.9 give

$$U^{x \otimes \tau(\mathcal{H})} = U_{13}^x: x \otimes \tau(\mathcal{H}) \otimes \rho \rightarrow \tau(x) \otimes \tau(\mathcal{H}) \otimes \rho.$$

Let $x \in \mathfrak{C}$. Then $U^x \in \mathfrak{C}(x \otimes \rho, \tau x \otimes \rho)$ is an intertwiner. So we may apply naturality to it. This and condition (2.8.2) give the commuting diagram of unitaries

$$\begin{array}{ccccc} \tau(x) \otimes \rho \otimes \rho & \xleftarrow{U_{12}^x} & x \otimes \rho \otimes \rho & \xrightarrow{U_{23}^\rho} & x \otimes \tau(\rho) \otimes \rho \\ \downarrow U_{23}^\rho = U^{\tau x \otimes \rho} & & \downarrow U^{x \otimes \rho} & & \downarrow U_{13}^x \\ \tau(x) \otimes \tau(\rho) \otimes \rho & \xleftarrow{U_{12}^x} & \tau(x \otimes \rho) \otimes \rho & \xlongequal{\quad} & \tau(x) \otimes \tau(\rho) \otimes \rho \end{array}$$

That is, $U_{12}^x U_{13}^x U_{23}^\rho = U_{23}^\rho U_{12}^x$. When we take $x = \rho$, this is the pentagon equation for U^ρ . For general x , it says that U^x is a right representation of U^ρ .

The naturality of U^x says that arrows $x_1 \rightarrow x_2$ in \mathfrak{C} are intertwiners $U^{x_1} \rightarrow U^{x_2}$. To prove that we have a fully faithful functor, we must show the converse. So let $a: \mathcal{H}^{x_1} \rightarrow \mathcal{H}^{x_2}$ be an intertwiner $U^{x_1} \rightarrow U^{x_2}$. Then we get an arrow

$$x_1 \otimes \rho \xrightarrow{U^{x_1}} \tau(x_1) \otimes \rho \xrightarrow{\tau(a) \otimes \text{id}_\rho} \tau(x_2) \otimes \rho \xrightarrow{(U^{x_2})^{-1}} x_2 \otimes \rho$$

Since a is an intertwiner, the forgetful functor maps this composite arrow to $a \otimes \text{id}_{\mathcal{H}^\rho}$. Since this operator comes from \mathfrak{C} , Assumption 2.3 ensures that a also comes from \mathfrak{C} . Thus any intertwiner comes from an arrow in \mathfrak{C} . This finishes the proof that the functor from \mathfrak{C} to the category of right representations of U^ρ is fully faithful. By construction, our functor intertwines the forgetful functors to \mathfrak{Hilb} .

The condition (2.8.2) says exactly that $U^{x_1 \otimes x_2}$ is the tensor product representation $U^{x_1} \otimes U^{x_2}$. Since we assumed \mathfrak{Fot} to map the associator and unit transformations in \mathfrak{C} to the usual ones in \mathfrak{Hilb} , the functor $x \mapsto U^x$ from \mathfrak{C} to the representation category of U^ρ is a strict tensor functor. \square

We have not found a “nice” characterisation when the functor $\mathfrak{C} \rightarrow \mathfrak{Rep}(U^\rho)$ is essentially surjective, that is, when every representation of U^ρ comes from an object of \mathfrak{C} .

Example 2.12. Let $\mathbb{W} \in \mathcal{U}(\mathcal{H} \otimes \mathcal{H})$ be a multiplicative unitary. A *left representation* of \mathbb{W} on a Hilbert space \mathcal{K} is a unitary $\hat{\mathbb{V}} \in \mathcal{U}(\mathcal{H} \otimes \mathcal{K})$ satisfying

$$\hat{\mathbb{V}}_{23} \mathbb{W}_{12} = \mathbb{W}_{12} \hat{\mathbb{V}}_{13} \hat{\mathbb{V}}_{23} \in \mathcal{U}(\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{K}).$$

The tensor product of two left representations $\hat{\mathbb{V}}^i \in \mathcal{U}(\mathcal{H} \otimes \mathcal{K}^i)$, $i = 1, 2$, is the left representation on $\mathcal{K}^1 \otimes \mathcal{K}^2$ defined by

$$\hat{\mathbb{V}}^1 \oplus \hat{\mathbb{V}}^2 := \hat{\mathbb{V}}_{13}^2 \hat{\mathbb{V}}_{12}^1 \in \mathcal{U}(\mathcal{H} \otimes \mathcal{K}^1 \otimes \mathcal{K}^2).$$

Left representations of \mathbb{W} also form a Hilbert space tensor category with the obvious forgetful functor and $\tau(\mathcal{H}) = (\mathcal{H}, 1)$. Actually, this tensor category is *isomorphic* to the category of right representations of the dual multiplicative unitary $\widehat{\mathbb{W}} = \Sigma \mathbb{W}^* \Sigma$: the isomorphism takes a left representation $\hat{\mathbb{V}} \in \mathcal{U}(\mathcal{H} \otimes \mathcal{K})$ to the right representation $\Sigma \hat{\mathbb{V}}^* \Sigma \in \mathcal{U}(\mathcal{H} \otimes \mathcal{K})$ of $\widehat{\mathbb{W}}$. Since $\widehat{\mathbb{W}}$ is a natural right absorber for right

representations of $\widehat{\mathbb{W}}$ by Example 2.10, the unitary \mathbb{W} , viewed as a left representation, is a natural *right* absorber in the tensor category of left representations of \mathbb{W} . The natural intertwiner is

$$\Sigma \hat{V}^* \Sigma: (\mathcal{K} \otimes \mathcal{H}, \hat{V} \oplus \mathbb{W}) \rightarrow (\mathcal{K} \otimes \mathcal{H}, 1_{\mathcal{K}} \oplus \mathbb{W}).$$

Next we want to prove that the multiplicative unitaries for two natural right absorbers of \mathfrak{C} are isomorphic in the category of multiplicative unitaries introduced in [6] and further studied in [2].

Proposition 2.13. *Let $(\rho, (U^x)_{x \in \mathfrak{C}})$ and $(\check{\rho}, (\check{U}^x)_{x \in \mathfrak{C}})$ be two natural right absorbers for \mathfrak{C} . Let $\mathcal{H} := \mathcal{H}^\rho$, $\check{\mathcal{H}} := \mathcal{H}^{\check{\rho}}$, $U := U^\rho \in \mathcal{U}(\mathcal{H} \otimes \mathcal{H})$, $\check{U} := U^{\check{\rho}} \in \mathcal{U}(\check{\mathcal{H}} \otimes \check{\mathcal{H}})$ be the corresponding multiplicative unitaries. The unitaries*

$$V := U^{\check{\rho}} \in \mathcal{U}(\check{\mathcal{H}} \otimes \mathcal{H}), \quad W := \check{U}^\rho \in \mathcal{U}(\mathcal{H} \otimes \check{\mathcal{H}})$$

satisfy the following pentagon-like equations:

$$\begin{aligned} U_{23}V_{12} &= V_{12}V_{13}U_{23}, & \check{U}_{23}W_{12} &= W_{12}W_{13}\check{U}_{23}, \\ V_{23}\check{U}_{12} &= \check{U}_{12}V_{13}V_{23}, & W_{23}U_{12} &= U_{12}W_{13}W_{23}, \\ V_{23}W_{12} &= W_{12}U_{13}V_{23}, & W_{23}V_{12} &= V_{12}\check{U}_{13}W_{23}. \end{aligned}$$

If the multiplicative unitaries U and \check{U} are manageable, then V and W give morphisms between the corresponding C^ -quantum groups that are inverse to each other in the category of C^* -quantum groups defined in [2].*

Proof. Our assumptions are symmetric in (ρ, U) and $(\check{\rho}, \check{U})$. When we exchange them, the equations in the first column become the corresponding ones in the second column. So it suffices to prove those in the first column. We already know that $V = U^{\check{\rho}}$ is a right representation of U , which gives the first equation. The other equations are proved similarly. For the second equation, we use the naturality of U for the intertwiner $\check{U}: \check{\rho} \otimes \check{\rho} \rightarrow \tau(\check{\rho}) \otimes \check{\rho}$ and rewrite $U^{\check{\rho} \otimes \check{\rho}} = U_{13}^{\check{\rho}} U_{23}^{\check{\rho}} = V_{13} V_{23}$ and $U^{\tau(\check{\rho}) \otimes \check{\rho}} = U_{23}^{\check{\rho}} = V_{23}$. For the third equation, we use the naturality of U for the intertwiner $W: \rho \otimes \check{\rho} \rightarrow \tau(\rho) \otimes \check{\rho}$ and rewrite $U^{\rho \otimes \check{\rho}} = U_{13}^\rho U_{23}^{\check{\rho}} = U_{13} V_{23}$ and $U^{\tau(\rho) \otimes \check{\rho}} = U_{23}^\rho = V_{23}$.

Morphisms of quantum groups are described in [2, Lemma 3.2]. The equations $U_{23}V_{12} = V_{12}V_{13}U_{23}$ and $V_{23}\check{U}_{12} = \check{U}_{12}V_{13}V_{23}$ say that V is a morphism from \check{U} to U . The equations $\check{U}_{23}W_{12} = W_{12}W_{13}\check{U}_{23}$ and $W_{23}U_{12} = U_{12}W_{13}W_{23}$ say that W is a morphism from U to \check{U} . The product of two morphisms is defined in [2, Definition 3.5] as the solution of a certain operator equation. The equation $V_{23}W_{12} = W_{12}U_{13}V_{23}$ says that the product of V and W is U . The equation $W_{23}V_{12} = V_{12}\check{U}_{13}W_{23}$ says that the product of W and V is \check{U} . Manageability is needed in [2] to ensure that the equation in [2, Definition 3.5] always has a solution. So manageability is needed to talk about a *category* of morphisms between multiplicative unitaries. \square

Example 2.14. Let (ρ, U) be a natural right absorber for \mathfrak{C} and let $y \in \mathfrak{C}$. Then $\check{\rho} = \rho \otimes y$ with $\check{U}^x := U^x \otimes \text{id}_y$ for all $x \in \mathfrak{C}$ is a natural right absorber as well. The corresponding multiplicative unitary is

$$(2.1) \quad \check{U}^{\rho \otimes y} = (U^{\rho \otimes y})_{123} = U_{13}^\rho U_{23}^y \in \mathcal{U}(\mathcal{H}^\rho \otimes \mathcal{H}^y \otimes \mathcal{H}^\rho \otimes \mathcal{H}^y).$$

Proposition 2.13 shows that both natural right absorbers give isomorphic multiplicative unitaries, provided both multiplicative unitaries involved are manageable, compare [5, Theorem 3.7].

We now extend the analysis above to describe functors between representation categories. Let \mathfrak{C}_1 and \mathfrak{C}_2 be Hilbert space tensor categories with natural right absorbers (ρ_1, U_1) and (ρ_2, U_2) , respectively. Let $\Phi: \mathfrak{C}_1 \rightarrow \mathfrak{C}_2$ be a strict tensor functor with $\mathfrak{F}\mathfrak{or}_2 \circ \Phi = \mathfrak{F}\mathfrak{or}_1$. If \mathfrak{C}_1 and \mathfrak{C}_2 are representation categories, then this means that Φ turns a representation of one sort into one of the other *on the same Hilbert space* in a natural way and preserving tensor products. Such a functor also satisfies $\Phi \circ \tau_1 = \tau_2$ by the argument in Remark 2.7.

Proposition 2.15. *Let $\Phi: \mathfrak{C}_1 \rightarrow \mathfrak{C}_2$ be a strict tensor functor with $\mathfrak{F}\mathfrak{or}_2 \circ \Phi = \mathfrak{F}\mathfrak{or}_1$. The unitary $V^\Phi := U_2^{\Phi(\rho_1)} \in \mathcal{U}(\mathcal{H}^{\rho_1} \otimes \mathcal{H}^{\rho_2})$ satisfies*

$$(U_2^{\rho_2})_{23} V_{12}^\Phi = V_{12}^\Phi V_{13}^\Phi (U_2^{\rho_2})_{23}, \quad V_{23}^\Phi (U_1^{\rho_1})_{12} = (U_1^{\rho_1})_{12} V_{13}^\Phi V_{23}^\Phi,$$

that is, V^Φ is a bicharacter from $U_1^{\rho_1}$ to $U_2^{\rho_2}$. Moreover, for any $x \in \mathfrak{C}_1$,

$$V_{23}^\Phi (U_1^x)_{12} = (U_1^x)_{12} (U_2^{\Phi(x)})_{13} V_{23}^\Phi \in \mathcal{U}(\mathcal{H}^x \otimes \mathcal{H}^{\rho_1} \otimes \mathcal{H}^{\rho_2}).$$

If the multiplicative unitary $U_1^{\rho_1}$ is manageable and $\mathfrak{C}_2 \cong \mathfrak{Rep}(U_2^{\rho_2})$, then the map from functors $\Phi: \mathfrak{C}_1 \rightarrow \mathfrak{C}_2$ as above to unitary bicharacters $V \in \mathcal{U}(\mathcal{H}^{\rho_1} \otimes \mathcal{H}^{\rho_2})$ from $U_1^{\rho_1}$ to $U_2^{\rho_2}$ is bijective. If the multiplicative unitaries $U_1^{\rho_1}$ and $U_2^{\rho_2}$ are both manageable, then V^Φ is a morphism between the corresponding C^ -quantum groups in the category defined in [2].*

Proof. The first two equations in the proposition say that V^Φ is a morphism of C^* -quantum groups as in [2, Lemma 3.2] provided the multiplicative unitaries $U_1^{\rho_1}$ and $U_2^{\rho_2}$ are manageable, so that they generate C^* -quantum groups. We already know that V^Φ is a right representation of $U_2^{\rho_2}$, which is the first equation. The second equation is the special case $x = \rho_1$ of the third one. The third equation says that the functor on representation categories induced by V^Φ is Φ , as expected. We prove this third equation by identifying

$$\begin{aligned} \Phi(x \otimes \rho_1) &= \Phi(x) \otimes \Phi(\rho_1), & U_2^{\Phi(x \otimes \rho_1)} &= (U_2^{\Phi(x)})_{13} V_{23}^\Phi, \\ \Phi(\tau(x) \otimes \rho_1) &= \tau(\Phi(x)) \otimes \Phi(\rho_1), & U_2^{\Phi(\tau(x) \otimes \rho_1)} &= (U_2^{\Phi(\rho_1)})_{23} = V_{23}^\Phi, \end{aligned}$$

and using the naturality of U_2 for the intertwiner $\Phi(U_1^x): \Phi(x \otimes \rho_1) \rightarrow \Phi(\tau(x) \otimes \rho_1)$. This gives $V_{23}^\Phi (U_1^x)_{12} = (U_1^x)_{12} (U_2^{\Phi(x)})_{13} V_{23}^\Phi$. The object $\Phi(x)$ of \mathfrak{C}_2 is determined uniquely by the corresponding representation $(U_2^{\Phi(x)})_{13}$ of $U_2^{\rho_2}$ by Proposition 2.11. This is given by $(U_2^{\Phi(x)})_{13} = (U_1^x)_{12}^* V_{23}^\Phi (U_1^x)_{12} (V_{23}^\Phi)^*$. The action of Φ on arrows is determined by the faithful forgetful functor to Hilbert spaces. So the unitary V^Φ determines the functor Φ .

Now assume that $U_1^{\rho_1}$ is manageable. Let $V \in \mathcal{U}(\mathcal{H}^{\rho_1} \otimes \mathcal{H}^{\rho_2})$ be a bicharacter. We lift it to a functor Ψ . Given $x \in \mathfrak{C}_1$, the operator $U_2^{\Phi(x)}$ must be such that

$$(U_2^{\Phi(x)})_{13} = (U_1^x)_{12}^* V_{23} (U_1^x)_{12} V_{23}^*.$$

We may argue as in the proof of [2, Lemma 3.6] to see that the operator on the right hand side of this equation is of the form $(U_2^{\Phi(x)})_{13}$ for some unitary $U_2^{\Phi(x)}$ and that the latter unitary is a representation of $U_2^{\rho_2}$. There is a unique functor $\Phi: \mathfrak{C}_1 \rightarrow \mathfrak{Rep}(U_2^{\rho_2})$ with $\mathfrak{F}\mathfrak{or} \circ \Phi = \mathfrak{F}\mathfrak{or}$ that sends $x \in \mathfrak{C}$ to this representation and acts identically on arrows, viewed as Hilbert space operators. This functor is a strict tensor functor as well. Any functor Φ is of this form for the corresponding bicharacter V^Φ . This gives the desired bijection. \square

Proposition 2.15 gives yet another equivalent characterisation of the quantum group morphisms of [2]: they are equivalent to strict tensor functors between the representation categories with $\mathfrak{F}\mathfrak{or} \circ \Phi = \mathfrak{F}\mathfrak{or}$. This result is similar in spirit to [2, Theorem 6.1], which uses coactions on C^* -algebras instead of representations.

2.1. Left and right absorbers. A natural left absorber in \mathfrak{C} is defined like a natural right absorber, but on the other side:

Definition 2.16. A *natural left absorber* in \mathfrak{C} is an object $\lambda \in \mathfrak{C}$ together with unitaries

$$U_\lambda^x: \lambda \otimes x \rightarrow \lambda \otimes \tau(x) \quad \text{for all } x \in \mathfrak{C}$$

with the following properties:

(2.16.1) the unitaries U_λ^x are natural in the sense that the following diagram commutes for any arrow $a: x_1 \rightarrow x_2$:

$$\begin{array}{ccc} \lambda \otimes x_1 & \xrightarrow[U_\lambda^{x_1}]{\cong} & \lambda \otimes \tau(x_1) \\ \text{id}_\lambda \otimes a \downarrow & & \downarrow \text{id}_\lambda \otimes a \\ \lambda \otimes x_2 & \xrightarrow[U_\lambda^{x_2}]{\cong} & \lambda \otimes \tau(x_2) \end{array}$$

(2.16.2) for $x_1, x_2 \in \mathfrak{C}$ the unitaries $U_\lambda^{x_1 \otimes x_2}$, $U_\lambda^{x_1}$ and $U_\lambda^{x_2}$ are related by the following commuting diagram:

$$\begin{array}{ccc} \lambda \otimes x_1 \otimes x_2 & \xrightarrow{U_\lambda^{x_1 \otimes x_2}} & \lambda \otimes \tau(x_1 \otimes x_2) \\ (U_\lambda^{x_1})_{12} \downarrow & & \parallel \\ \lambda \otimes \tau(x_1) \otimes x_2 & \xrightarrow{(U_\lambda^{x_2})_{13}} & \lambda \otimes \tau(x_1) \otimes \tau(x_2) \end{array}$$

The analogue of Lemma 2.9 holds for natural left absorbers as well, that is, $U_\lambda^{\tau(\mathcal{H})} = \text{id}_{\tau(\mathcal{H}) \otimes \lambda}$ for any Hilbert space \mathcal{H} .

Let \mathbb{W} be a multiplicative unitary. Then the categories of left and of right representations of \mathbb{W} have a canonical natural *right* absorber by Examples 2.10 and 2.12. It is unclear, in general, whether they have a natural left absorber as well. The only construction of left absorbers that we know uses the contragradient operation to turn a right into a left absorber. For contragredients to exist, we assume \mathbb{W} to be manageable. We work with right representations of \mathbb{W} to be definite. The contragradient of a representation \mathbb{U} on a Hilbert space \mathcal{H} is a representation $\widetilde{\mathbb{U}}$ on the complex-conjugate Hilbert space $\overline{\mathcal{H}}$. The contragradient construction becomes a covariant functor $\mathfrak{Rep}(\mathbb{W}) \rightarrow \mathfrak{Rep}(\mathbb{W})$ when we map an intertwiner $a: \mathcal{H}_1 \rightarrow \mathcal{H}_2$ to $\overline{a}: \overline{\mathcal{H}}_1 \rightarrow \overline{\mathcal{H}}_2$. This is not quite a W^* -functor because it is conjugate-linear, not linear. The contragradient of a trivial representation remains trivial. The contragradient operation is involutive, that is, $\widetilde{\widetilde{\mathbb{U}}} = \mathbb{U}$ for representations and $\widetilde{\overline{a}} = a$ for intertwiners. It reverses the order of tensor factors: the flip operator $\Sigma: \overline{\mathcal{H}}_1 \otimes \overline{\mathcal{H}}_2 \rightarrow \overline{\mathcal{H}}_2 \otimes \overline{\mathcal{H}}_1 \cong \overline{\mathcal{H}}_2 \otimes \overline{\mathcal{H}}_1$ intertwines $\widetilde{\mathbb{U}}_1 \oplus \widetilde{\mathbb{U}}_2$ with the contragradient of $\mathbb{U}_1 \oplus \mathbb{U}_2$, see [9, Section 3].

Let $(\rho, (U^x)_{x \in \mathfrak{Rep}(\mathbb{W})})$ be a natural right absorber for $\mathfrak{Rep}(\mathbb{W})$. For instance, we may take the canonical one described in Example 2.10. Let $\lambda := \widetilde{\rho}$ be the contragradient of ρ , so $\widetilde{\lambda} = \rho$. Let $U_\lambda^x: \lambda \otimes x \rightarrow \tau(\lambda) \otimes x$ for $x \in \mathfrak{Rep}(\mathbb{W})$ be the composite unitary intertwiner

$$\lambda \otimes x \cong \widetilde{\widetilde{\lambda}} \otimes x \xrightarrow{\widetilde{\widetilde{U^x}}} \widetilde{\widetilde{\tau(\lambda)}} \otimes \widetilde{\widetilde{x}} \xrightarrow{\widetilde{\widetilde{U^x}}} \tau(\widetilde{\widetilde{\lambda}}) \otimes \widetilde{\widetilde{x}} \rightarrow \tau(\widetilde{\widetilde{\lambda}}) \otimes \widetilde{\widetilde{x}} \rightarrow \lambda \otimes \tau(x).$$

Routine computations show that $(\lambda, (U_\lambda^x))$ is a natural left absorber if $(\rho, (U^x))$ is a natural right absorber. This proves the following:

Proposition 2.17. *Let \mathbb{W} be a multiplicative unitary. If \mathbb{W} is manageable, then its tensor category of representations $\mathfrak{Rep}(\mathbb{W})$ contains both a natural right absorber and a natural left absorber.*

If \mathfrak{C} has both a right absorber ρ and a left absorber λ , then

$$\tau(\lambda) \otimes \rho \cong \lambda \otimes \rho \cong \lambda \otimes \tau(\rho).$$

That is, the direct sums of infinitely many copies of λ and ρ are isomorphic. This common direct sum is both a left and a right absorber, and its isomorphism class does not depend on the choice of λ or ρ . These observations go back already to [7], and they need only absorption, without any naturality. We are going to use the uniqueness of two-sided absorbers to prove that any isomorphism between the representation categories of two C^* -quantum groups comes from an isomorphism of Hopf $*$ -algebras. First we need a preparatory result, which would really belong into [2], but was not proved there.

Theorem 2.18. *The isomorphisms in the category of C^* -quantum groups defined in [2] are the same as the Hopf $*$ -isomorphisms of the underlying C^* -bialgebras.*

Proof. It is trivial that a Hopf $*$ -isomorphism induces an isomorphism in the category of [2]. Conversely, an isomorphism between two C^* -quantum groups (C_i, Δ_{C_i}) , $i = 1, 2$, in the category of [2] only gives a Hopf $*$ -isomorphism between their universal dual quantum groups $\hat{C}_1^u \cong \hat{C}_2^u$ (or $C_1^u \cong C_2^u$, but we shall use the dual isomorphism below). The Hopf $*$ -isomorphism $\hat{C}_1^u \cong \hat{C}_2^u$ induces an isomorphism between the representation categories of (C_1, Δ_{C_1}) and (C_2, Δ_{C_2}) .

Let $\mathbb{W}_i \in \mathcal{U}(\mathcal{H}_i \otimes \mathcal{H}_i)$, $i = 1, 2$, be manageable multiplicative unitaries that generate (C_i, Δ_{C_i}) . We view \mathbb{W}_i as a right representation of (C_i, Δ_{C_i}) on \mathcal{H}_i . The representation of \hat{C}_i^u associated to \mathbb{W}_i descends to a faithful representation of \hat{C}_i : this is the standard construction of $\hat{C}_i \subseteq \mathbb{B}(\mathcal{H}_i)$ from a multiplicative unitary in [9]. Thus we have to prove that the representations of $\hat{C}_1^u \cong \hat{C}_2^u$ associated to \mathbb{W}_1 and \mathbb{W}_2 have the same kernel. Since our multiplicative unitaries are manageable, the representation category

$$\mathfrak{C} := \mathfrak{Rep}(\mathbb{W}_1) \cong \mathfrak{Rep}((C_1, \Delta_{C_1})) \cong \mathfrak{Rep}((C_2, \Delta_{C_2})) \cong \mathfrak{Rep}(\mathbb{W}_2)$$

contains both a natural left and a natural right absorber by Proposition 2.17. Both \mathbb{W}_1 and \mathbb{W}_2 are natural right absorbers. By the remarks above, the direct sums $(\mathbb{W}_1)^\infty$ and $(\mathbb{W}_2)^\infty$ of infinitely many copies of \mathbb{W}_1 and \mathbb{W}_2 are isomorphic objects of \mathfrak{C} because they are both isomorphic to the direct sum of infinitely many copies of a natural left absorber. Therefore, the representations of \hat{C}_1^u associated to $(\mathbb{W}_1)^\infty$ and $(\mathbb{W}_2)^\infty$ have the same kernel. Then the representations of \hat{C}_1^u associated to \mathbb{W}_1 and \mathbb{W}_2 also have the same kernel. Thus our Hopf $*$ -isomorphism $\hat{C}_1^u \cong \hat{C}_2^u$ descends to a Hopf $*$ -isomorphism $\hat{C}_1 \cong \hat{C}_2$. This implies a Hopf $*$ -isomorphism $C_1 \cong C_2$. \square

Corollary 2.19. *A C^* -quantum group (C, Δ_C) is determined uniquely by its tensor category $\mathfrak{Rep}(C, \Delta_C)$ of representations with the forgetful functor to \mathfrak{Hilb} .*

Proof. Let \mathbb{W} be a manageable multiplicative unitary that generates the C^* -quantum group (C, Δ_C) . Then a representation of (C, Δ_C) is equivalent to one of \mathbb{W} on the same Hilbert space, so that we may identify $\mathfrak{Rep}(C, \Delta_C) \cong \mathfrak{Rep}(\mathbb{W})$. Now let (C_1, Δ_{C_1}) and (C_2, Δ_{C_2}) be two C^* -quantum groups. Let \mathbb{W}_1 and \mathbb{W}_2 be manageable multiplicative unitaries that generate them. These are natural right absorbers by Example 2.10. If $\mathfrak{Rep}(C_1, \Delta_{C_1}) \cong \mathfrak{Rep}(C_2, \Delta_{C_2})$, then Proposition 2.13 shows that \mathbb{W}_1 and \mathbb{W}_2 are isomorphic in the category of [2]. Theorem 2.18 shows that this isomorphism is a Hopf $*$ -isomorphism. \square

Proposition 2.11 has a variant for natural left absorbers. Let λ and $(U^x)_{x \in \mathfrak{C}}$ be a natural left absorber for \mathfrak{C} . For $x \in \mathfrak{C}$, let $\mathcal{H}^x := \mathfrak{For}(x)$, and write U^x for

$\mathfrak{For}(U^x) \in \mathcal{U}(\mathcal{H}^\lambda \otimes \mathcal{H}^x)$. Then U^λ is an “antimultiplicative” unitary:

$$U_{12}^\lambda U_{23}^\lambda = U_{23}^\lambda U_{13}^\lambda U_{12}^\lambda.$$

Moreover, U^x for $x \in \mathfrak{C}$ is a left representation of U^λ :

$$U_{23}^x U_{13}^\lambda U_{12}^\lambda = U_{12}^\lambda U_{23}^x.$$

We define a tensor product for representations of U^λ by

$$U \oplus V := V_{13} U_{12}.$$

The map $x \mapsto U^x$ gives a fully faithful, strict tensor functor from \mathfrak{C} to $\mathfrak{Rep}(U^\lambda)$, which intertwines the forgetful functors from \mathfrak{C} and $\mathfrak{Rep}(U^\lambda)$ to \mathfrak{Hilb} .

Similarly, there is an analogue of Proposition 2.13, saying that the antimultiplicative unitaries $\mathcal{H} := \mathcal{H}^\lambda$, $\check{\mathcal{H}} := \mathcal{H}^{\check{\lambda}}$, $U := U^\lambda$, $\check{U} := \check{U}^{\check{\lambda}}$ associated to two natural left absorbers $(\lambda, (U^x)_{x \in \mathfrak{C}})$ and $(\check{\lambda}, (\check{U}^x)_{x \in \mathfrak{C}})$ are “isomorphic” in a suitable sense. Namely, the unitaries

$$V := U^{\check{\lambda}} \in \mathcal{U}(\mathcal{H} \otimes \check{\mathcal{H}}), \quad W := \check{U}^\lambda \in \mathcal{U}(\check{\mathcal{H}} \otimes \mathcal{H})$$

satisfy the following pentagon-like equations:

$$\begin{aligned} U_{12} V_{23} &= V_{23} V_{13} U_{12}, & \check{U}_{12} W_{23} &= W_{23} W_{13} \check{U}_{12}, \\ V_{12} \check{U}_{23} &= \check{U}_{23} V_{13} V_{12}, & W_{12} U_{23} &= U_{23} W_{13} W_{12}, \\ V_{12} W_{23} &= W_{23} U_{13} V_{12}, & W_{12} V_{23} &= V_{23} \check{U}_{13} W_{12}. \end{aligned}$$

It is also interesting to apply the same technique to a tensor category with a natural right absorber $(\rho, (U^x)_{x \in \mathfrak{C}})$ and a natural left absorber $(\lambda, (\check{U}^x)_{x \in \mathfrak{C}})$. Let $\mathcal{H} := \mathcal{H}^\rho$, $\check{\mathcal{H}} := \mathcal{H}^\lambda$, $U := U^\rho$, $\check{U} := \check{U}^\lambda$ be the associated multiplicative and antimultiplicative unitaries. Define

$$V := U^\lambda \in \mathcal{U}(\check{\mathcal{H}} \otimes \mathcal{H}), \quad W := \check{U}^\rho \in \mathcal{U}(\check{\mathcal{H}} \otimes \mathcal{H}).$$

These unitaries satisfy the following pentagon-like equations:

$$\begin{aligned} \check{U}_{12} V_{23} &= V_{23} V_{13} \check{U}_{12}, & \check{U}_{12} W_{23} &= W_{23} W_{13} \check{U}_{12}, \\ U_{23} V_{12} &= V_{12} V_{13} U_{23}, & U_{23} W_{12} &= W_{12} W_{13} U_{23}, \\ V_{13} W_{12} &= W_{12} V_{13} U_{23}, & W_{13} V_{23} &= V_{23} W_{13} \check{U}_{13}. \end{aligned}$$

The proofs are similar to those in Proposition 2.13. In addition, let x be any object of \mathfrak{C} . Naturality of \check{U} with respect to the intertwiner $U^x: x \otimes \rho \rightarrow \tau(x) \otimes \rho$ gives

$$(2.2) \quad \check{U}_{12}^x = (\check{U}_{13}^\rho)^* (U_{23}^x)^* \check{U}_{13}^\rho U_{23}^x = W_{13}^* (U_{23}^x)^* W_{13} U_{23}^x.$$

Naturality of U with respect to the intertwiner $\check{U}^x: \lambda \otimes x \rightarrow \lambda \otimes \tau(x)$ gives

$$(2.3) \quad U_{23}^x = (U_{23}^\lambda)^* (\check{U}_{12}^x)^* U_{23}^\lambda \check{U}_{12}^x = V_{23}^* (\check{U}_{12}^x)^* V_{23} \check{U}_{12}^x.$$

Here U^x and \check{U}^x are the representations of U and \check{U} associated to x , respectively. So these determine each other. If $\mathfrak{C} = \mathfrak{Rep}(U)$ for a manageable multiplicative unitary U and \check{U} comes from its contragradient as above, then also $\mathfrak{C} = \mathfrak{Rep}(\check{U})$. So for a given representation U^x of U , there is a unique representation \check{U}^x of \check{U} satisfying (2.2). And for a given representation \check{U}^x of \check{U} , there is a unique representation U^x of U satisfying (2.3).

Multiplicative and antimultiplicative unitaries are closely related to the Heisenberg and anti-Heisenberg pairs studied in [3]. By definition, a Heisenberg pair for a C^* -quantum group (C, Δ_C) is a pair of representations $(\pi, \hat{\pi})$ of (C, \hat{C}) such that $(\hat{\pi} \otimes \pi)W$ for the reduced bicharacter $W \in \mathcal{U}(\hat{C} \otimes C)$ is a multiplicative unitary. And an anti-Heisenberg pair is a pair of representations $(\sigma, \hat{\sigma})$ of (C, \hat{C}) such that $(\hat{\sigma} \otimes \sigma)W$ is an antimultiplicative unitary.

3. REPRESENTATIONS OF BRAIDED MULTIPLICATIVE UNITARIES

Let \mathcal{H} and \mathcal{L} be Hilbert spaces and let $\mathbb{W} \in \mathcal{U}(\mathcal{H} \otimes \mathcal{H})$ be a multiplicative unitary. Let

$$\mathbb{U} \in \mathcal{U}(\mathcal{L} \otimes \mathcal{H}), \quad \hat{\mathbb{V}} \in \mathcal{U}(\mathcal{H} \otimes \mathcal{L}), \quad \mathbb{F} \in \mathcal{U}(\mathcal{L} \otimes \mathcal{L}),$$

be a braided multiplicative unitary over \mathbb{W} (see [5]). We are first going to define a tensor category $\mathfrak{Rep}(\mathbb{W}, \mathbb{U}, \hat{\mathbb{V}}, \mathbb{F})$ of right representations.

Definition 3.1. A (right) *representation* of $(\mathbb{W}, \mathbb{U}, \hat{\mathbb{V}}, \mathbb{F})$ is a triple $(\mathcal{K}, \mathbb{S}, \mathbb{T})$, where \mathcal{K} is a Hilbert space, $\mathbb{S} \in \mathcal{U}(\mathcal{K} \otimes \mathcal{H})$ is a right representation of \mathbb{W} on \mathcal{K} , that is,

$$(3.1) \quad \mathbb{W}_{23} \mathbb{S}_{12} = \mathbb{S}_{12} \mathbb{S}_{13} \mathbb{W}_{23} \quad \text{in } \mathcal{U}(\mathcal{K} \otimes \mathcal{H} \otimes \mathcal{H}),$$

and $\mathbb{T} \in \mathcal{U}(\mathcal{K} \otimes \mathcal{L})$ is equivariant with respect to the tensor product representation $\mathbb{S} \otimes \mathbb{U}$ of \mathbb{W} ,

$$(3.2) \quad \mathbb{S}_{13} \mathbb{U}_{23} \mathbb{T}_{12} = \mathbb{T}_{12} \mathbb{S}_{13} \mathbb{U}_{23} \quad \text{in } \mathcal{U}(\mathcal{K} \otimes \mathcal{L} \otimes \mathcal{H}),$$

and satisfies the (top-braided) representation condition

$$(3.3) \quad \mathbb{F}_{23} \mathbb{T}_{12} = \mathbb{T}_{12} (\mathcal{L} \times \mathcal{L})_{23} \mathbb{T}_{12} (\mathcal{L} \times \mathcal{L})_{23} \mathbb{F}_{23} \quad \text{in } \mathcal{U}(\mathcal{K} \otimes \mathcal{L} \otimes \mathcal{L}).$$

The braiding in (3.3) is the same as in the top-braided pentagon equation for \mathbb{F} . Hence $(\mathcal{L}, \mathbb{U}, \mathbb{F})$ is an example of such a right representation.

A morphism $(\mathcal{K}^1, \mathbb{S}^1, \mathbb{T}^1) \rightarrow (\mathcal{K}^2, \mathbb{S}^2, \mathbb{T}^2)$ is a bounded operator $a: \mathcal{K}^1 \rightarrow \mathcal{K}^2$ that intertwines both representations, that is, $a_1 \circ \mathbb{S}^1 = \mathbb{S}^2 \circ a_1$ and $a_1 \circ \mathbb{T}^1 = \mathbb{T}^2 \circ a_1$. This turns the representations of $(\mathbb{W}, \mathbb{U}, \hat{\mathbb{V}}, \mathbb{F})$ into a \mathbf{W}^* -category $\mathfrak{Rep}(\mathbb{W}, \mathbb{U}, \hat{\mathbb{V}}, \mathbb{F})$. Forgetting both representations \mathbb{S} and \mathbb{T} gives the forgetful functor to Hilbert spaces. The functor τ maps $\mathcal{K} \mapsto (\mathcal{K}, 1, 1)$. If the identity map on \mathcal{K} is an intertwiner $(\mathcal{K}, \mathbb{S}^1, \mathbb{T}^1) \rightarrow (\mathcal{K}, \mathbb{S}^2, \mathbb{T}^2)$, then $\mathbb{S}^1 = \mathbb{S}^2$ and $\mathbb{T}^1 = \mathbb{T}^2$. So Assumption 2.2 is satisfied.

We define a tensor product operation \oplus on $\mathfrak{Rep}(\mathbb{W}, \mathbb{U}, \hat{\mathbb{V}}, \mathbb{F})$ by

$$(\mathcal{K}^1, \mathbb{S}^1, \mathbb{T}^1) \oplus (\mathcal{K}^2, \mathbb{S}^2, \mathbb{T}^2) := (\mathcal{K}^1 \otimes \mathcal{K}^2, \mathbb{S}^1 \oplus \mathbb{S}^2, \mathbb{T}^1 \oplus \mathbb{T}^2)$$

with

$$\mathbb{S}^1 \oplus \mathbb{S}^2 = \mathbb{S}_{13}^1 \mathbb{S}_{23}^2 \in \mathcal{U}(\mathcal{K}^1 \otimes \mathcal{K}^2 \otimes \mathcal{H}),$$

$$\mathbb{T}^1 \oplus \mathbb{T}^2 = (\mathcal{L} \times \mathcal{K}^2)_{23} \mathbb{T}_{12}^1 (\mathcal{K}^2 \times \mathcal{L})_{23} \mathbb{T}_{23}^2 \in \mathcal{U}(\mathcal{K}^1 \otimes \mathcal{K}^2 \otimes \mathcal{L}).$$

The braiding operators $\mathcal{L} \times \mathcal{K}^2_{23}$ and $\mathcal{K}^2 \times \mathcal{L}$ use only the representations \mathbb{S} on \mathcal{K}^2 and $\hat{\mathbb{V}}$ on \mathcal{L} and therefore make sense. In contrast, $\mathcal{K}^2 \times \mathcal{L}$ and $\mathcal{L} \times \mathcal{K}^2$ would be defined if we had a left representation of \mathbb{W} on \mathcal{K}^2 instead of a right one.

Lemma 3.2. *The above definitions turn $\mathfrak{Rep}(\mathbb{W}, \mathbb{U}, \hat{\mathbb{V}}, \mathbb{F})$ into a Hilbert space tensor category.*

Proof. First, we ought to check that the tensor product above is well-defined, that is, gives representations again. We check associativity of the tensor product first because we want to use it to prove that the tensor product is again a representation. Let $\mathbb{S}^i \in \mathcal{U}(\mathcal{K}^i \otimes \mathcal{H})$ and $\mathbb{T}^i \in \mathcal{U}(\mathcal{K}^i \otimes \mathcal{L})$ for $i = 1, 2, 3$ be corepresentations of $\mathbb{W}, \mathbb{U}, \hat{\mathbb{V}}, \mathbb{F}$. The definition of $\mathbb{T} \oplus \mathbb{T}'$ makes sense for any \mathbb{W} -equivariant unitary operators \mathbb{T}, \mathbb{T}' . Thus both $(\mathbb{T}^1 \oplus \mathbb{T}^2) \oplus \mathbb{T}^3$ and $\mathbb{T}^1 \oplus (\mathbb{T}^2 \oplus \mathbb{T}^3)$ are defined even if we do not yet know that $\mathbb{T}^1 \oplus \mathbb{T}^2$ and $\mathbb{T}^2 \oplus \mathbb{T}^3$ give representations again. We claim that both $(\mathbb{T}^1 \oplus \mathbb{T}^2) \oplus \mathbb{T}^3$ and $\mathbb{T}^1 \oplus (\mathbb{T}^2 \oplus \mathbb{T}^3)$ are equal to the \mathbb{W} -equivariant unitary

$$(3.4) \quad (\mathcal{L} \times \mathcal{K}^2 \otimes \mathcal{K}^3)_{234} \mathbb{T}_{12}^1 (\mathcal{K}^2 \otimes \mathcal{K}^3 \times \mathcal{L})_{234} (\mathcal{L} \times \mathcal{K}^3)_{34} \mathbb{T}_{23}^2 (\mathcal{K}^3 \times \mathcal{L})_{34} \mathbb{T}_{34}^3$$

in $\mathcal{U}(\mathcal{K}^1 \otimes \mathcal{K}^2 \otimes \mathcal{K}^3 \otimes \mathcal{L})$. The operators $\mathcal{L} \times \mathcal{K}^i: \mathcal{L} \otimes \mathcal{K}^i \rightarrow \mathcal{K}^i \otimes \mathcal{L}$ are defined by $\mathcal{L} \times \mathcal{K}^i := Z^i \Sigma$, where $Z^i \in \mathcal{U}(\mathcal{K}^i \otimes \mathcal{L})$ satisfies

$$(3.5) \quad Z_{13}^i = \hat{\mathbb{V}}_{23} (\mathbb{S}_{12}^i)^* \hat{\mathbb{V}}_{23}^* \mathbb{S}_{12}^i \quad \text{in } \mathcal{U}(\mathcal{K}^i \otimes \mathcal{H} \otimes \mathcal{L})$$

for $i = 1, 2, 3$. And $\mathcal{L} \times^{\mathcal{K}^1 \otimes \mathcal{K}^2} = Z^{12} \Sigma_{23} \Sigma_{12}$, where $Z^{12} \in \mathcal{U}(\mathcal{K}^1 \otimes \mathcal{K}^2 \otimes \mathcal{L})$ satisfies

$$(3.6) \quad Z_{124}^{12} = \hat{\mathbb{V}}_{34}(\mathbb{S}^1 \oplus \mathbb{S}^2)_{123}^* \hat{\mathbb{V}}_{34}^*(\mathbb{S}^1 \oplus \mathbb{S}^2)_{123} \quad \text{in } \mathcal{U}(\mathcal{K}^1 \otimes \mathcal{K}^2 \otimes \mathcal{H} \otimes \mathcal{L}).$$

This equation gives $Z^{12} = Z_{23}^2 Z_{13}^1$ when we plug in the definition of \oplus and eliminate \mathbb{S}^1 , \mathbb{S}^2 and $\hat{\mathbb{V}}$ using (3.5). Therefore,

$$\mathcal{L} \times^{\mathcal{K}^1 \otimes \mathcal{K}^2} = Z^{12} \Sigma_{23} \Sigma_{12} = Z_{23}^2 Z_{13}^1 \Sigma_{23} \Sigma_{12} = Z_{23}^2 \Sigma_{23} Z_{12}^1 \Sigma_{12} = \mathcal{L} \times^{\mathcal{K}^2}_{23} \mathcal{L} \times^{\mathcal{K}^1}_{12}.$$

Similarly, $\mathcal{L} \times^{\mathcal{K}^2 \otimes \mathcal{K}^3} = \mathcal{L} \times^{\mathcal{K}^3}_{23} \mathcal{L} \times^{\mathcal{K}^2}_{12}$. Now both $\mathbb{T}^1 \oplus (\mathbb{T}^2 \oplus \mathbb{T}^3)$ and $(\mathbb{T}^1 \oplus \mathbb{T}^2) \oplus \mathbb{T}^3$ and the expression in (3.4) are equal because they all simplify to

$$\mathcal{L} \times^{\mathcal{K}^3}_{34} \mathcal{L} \times^{\mathcal{K}^2}_{23} \mathbb{T}_{12}^1 \mathcal{L} \times^{\mathcal{K}^2}_{23} \mathbb{T}_{23}^2 \mathcal{L} \times^{\mathcal{K}^3}_{34} \mathbb{T}_{34}^3.$$

Next, we check that the tensor product of two representations is again a representation. The proof will also help to construct a natural right absorber later. We claim that an operator $\mathbb{T} \in \mathcal{U}(\mathcal{K} \otimes \mathcal{L})$ together with $(\mathcal{K}, \mathbb{S}) \in \mathfrak{Rep}(\mathbb{W})$ gives a representation if and only if \mathbb{T} is an intertwiner

$$(\mathcal{K} \otimes \mathcal{L}, \mathbb{S} \oplus \mathbb{U}, \mathbb{T} \oplus \mathbb{F}) \rightarrow (\mathcal{K} \otimes \mathcal{L}, \mathbb{S} \oplus \mathbb{U}, 1 \oplus \mathbb{F}).$$

Indeed, being such an intertwiner means being equivariant with respect to $\mathbb{S} \oplus \mathbb{U}$ and intertwining $\mathbb{T} \oplus \mathbb{F} = (\mathcal{L} \times^{\mathcal{K}^2})_{23} \mathbb{T}_{12} (\mathcal{K}^2 \times^{\mathcal{L}})_{23} \mathbb{F}_{23}$ with $1 \oplus \mathbb{F} = \mathbb{F}_{23}$. The latter is exactly our representation condition. Assume we know this for $\mathbb{T}^1 \in \mathcal{U}(\mathcal{K}^1 \otimes \mathcal{L})$ and $\mathbb{T}^2 \in \mathcal{U}(\mathcal{K}^2 \otimes \mathcal{L})$. Since \mathbb{T}_{23}^2 is equivariant, when we conjugate it with the braiding operator $(\mathcal{L} \times^{\mathcal{K}^2 \otimes \mathcal{L}})_{234}$ on $\mathcal{K}^1 \otimes \mathcal{K}^2 \otimes \mathcal{L} \otimes \mathcal{L}$, then we merely transfer it to \mathbb{T}_{34}^2 , which commutes with \mathbb{T}_{12}^1 . Thus (3.4) shows that \mathbb{T}_{23}^2 is also an intertwining operator

$$(\mathcal{K}^1 \otimes \mathcal{K}^2 \otimes \mathcal{L}, \mathbb{S}^1 \oplus \mathbb{S}^2 \oplus \mathbb{U}, \mathbb{T}^1 \oplus \mathbb{T}^2 \oplus \mathbb{F}) \rightarrow (\mathcal{K}^1 \otimes \mathcal{K}^2 \otimes \mathcal{L}, \mathbb{S}^1 \oplus \mathbb{S}^2 \oplus \mathbb{U}, \mathbb{T}^1 \oplus 1 \oplus \mathbb{F}).$$

Similarly, the braiding operator $\mathcal{K}^2 \times^{\mathcal{L}}$ gives an intertwiner

$$(3.7) \quad (\mathcal{K}^1 \otimes \mathcal{K}^2 \otimes \mathcal{L}, \mathbb{S}^1 \oplus \mathbb{S}^2 \oplus \mathbb{U}, \mathbb{T}^1 \oplus 1 \oplus \mathbb{F}) \xrightarrow{\mathcal{K}^2 \times^{\mathcal{L}}_{23}} (\mathcal{K}^1 \otimes \mathcal{L} \otimes \mathcal{K}^2, \mathbb{S}^1 \oplus \mathbb{U} \oplus \mathbb{S}^2, \mathbb{T}^1 \oplus \mathbb{F} \oplus 1).$$

Now the operator \mathbb{T}_{12}^1 is an intertwiner

$$(\mathcal{K}^1 \otimes \mathcal{L} \otimes \mathcal{K}^2, \mathbb{S}^1 \oplus \mathbb{U} \oplus \mathbb{S}^2, \mathbb{T}^1 \oplus \mathbb{F} \oplus 1) \rightarrow (\mathcal{K}^1 \otimes \mathcal{L} \otimes \mathcal{K}^2, \mathbb{S}^1 \oplus \mathbb{U} \oplus \mathbb{S}^2, 1 \oplus \mathbb{F} \oplus 1).$$

The unitary $\mathcal{L} \times^{\mathcal{K}^2}$ gives an intertwiner from the last representation back to

$$(\mathcal{K}^1 \otimes \mathcal{K}^2 \otimes \mathcal{L}, \mathbb{S}^1 \oplus \mathbb{S}^2 \oplus \mathbb{U}, 1 \oplus 1 \oplus \mathbb{F}).$$

Hence $\mathbb{T}^1 \oplus \mathbb{T}^2$ has the expected intertwining property to be a representation.

Now we check Assumption 2.3. Let $a \in \mathbb{B}(\mathcal{K}^1)$ be such that $a_1 \in \mathcal{U}(\mathcal{K}^1 \otimes \mathcal{K}^2)$ is an intertwiner for the tensor product representation. Since a commutes with \mathbb{S}_{23}^2 , the equivariance with respect to $\mathbb{S}^1 \oplus \mathbb{S}^2$ gives that a is \mathbb{S}^1 -equivariant. Since a_1 commutes with $\mathbb{T}^1 \oplus \mathbb{T}^2$, $(\mathcal{L} \times^{\mathcal{K}^2 \otimes \mathcal{K}^3})_{234}$, and \mathbb{T}_{23}^2 , it follows that a is an intertwiner for \mathbb{T}_{12}^1 as well.

Finally, the functors $\mathfrak{F} \circ \tau$ and τ are strict tensor functors by definition, and $\tau(\mathbb{C})$ with the canonical unit transformations is indeed a tensor unit. \square

Proposition 3.3. *The representation*

$$\rho = (\mathcal{H} \otimes \mathcal{L}, \mathbb{W} \oplus \mathbb{U}, 1 \oplus \mathbb{F})$$

is a natural right absorber for the tensor category $\mathfrak{Rep}(\mathbb{W}, \mathbb{U}, \hat{\mathbb{V}}, \mathbb{F})$.

Proof. We must construct an intertwiner $A^x: x \otimes \rho \rightarrow \tau(x) \otimes \rho$ for any representation $x = (\mathcal{K}, \mathbb{S}, \mathbb{T})$ of $(\mathbb{W}, \mathbb{U}, \hat{\mathbb{V}}, \mathbb{F})$. We claim that the composite operator

$$\begin{aligned} (\mathcal{K} \otimes \mathcal{H} \otimes \mathcal{L}, \mathbb{S} \oplus \mathbb{W} \oplus \mathbb{U}, \mathbb{T} \oplus 1_{\mathcal{H}} \oplus \mathbb{F}) &\xrightarrow{\mathbb{T} \oplus 1_{\mathcal{H}}} (\mathcal{K} \otimes \mathcal{H} \otimes \mathcal{L}, \mathbb{S} \oplus \mathbb{W} \oplus \mathbb{U}, 1_{\mathcal{K}} \oplus 1_{\mathcal{H}} \oplus \mathbb{F}) \\ &\xrightarrow{\mathbb{S}_{12}} (\mathcal{K} \otimes \mathcal{H} \otimes \mathcal{L}, 1_{\mathcal{K}} \oplus \mathbb{W} \oplus \mathbb{U}, 1_{\mathcal{K}} \oplus 1_{\mathcal{H}} \oplus \mathbb{F}) \end{aligned}$$

has the properties required in Definition 2.8. The triple $(\mathcal{K}, \mathbb{S}, 1)$ is a representation of $(\mathbb{W}, \mathbb{U}, \hat{\mathbb{V}}, \mathbb{F})$ for any right representation \mathbb{S} of \mathbb{W} , and a map between representations of this form is an intertwiner if and only if it is an intertwiner for the representations of \mathbb{W} . In particular, the second map \mathbb{S}_{12} above is an intertwiner, see Example 2.10. Moreover, since there are representations $(\mathcal{H}, \mathbb{W}, 1)$ and $x \otimes (\mathcal{H}, \mathbb{W}, 1) = (\mathcal{K} \otimes \mathcal{H}, \mathbb{S} \oplus \mathbb{W}, \mathbb{F} \oplus 1)$, the map $\mathbb{T} \oplus 1_{\mathcal{H}}$ above is an intertwiner as well, see the proof of Lemma 3.2. Thus the composite map is an intertwiner $x \otimes \rho \rightarrow \tau(x) \otimes \rho$ as needed. These two operators and their composite are natural by construction, that is, (2.8.1) holds. We check condition (2.8.2). Let $(\mathcal{K}^i, \mathbb{S}^i, \mathbb{T}^i)$ be representations of $(\mathbb{W}, \mathbb{U}, \hat{\mathbb{V}}, \mathbb{F})$. We shall use the diagram in Figure 1. This

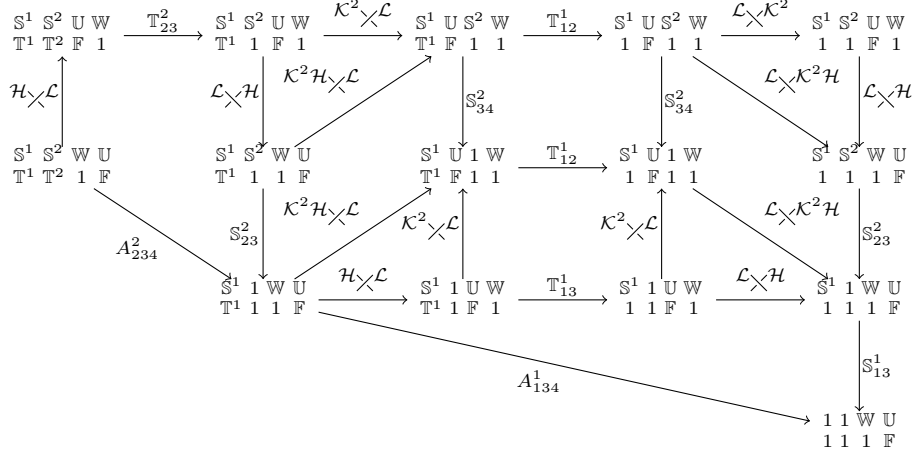


FIGURE 1. Commuting diagram in $\mathfrak{Rep}(\mathbb{W}, \mathbb{U}, \hat{\mathbb{V}}, \mathbb{F})$ that proves the condition (2.8.2).

diagram uses some short-hand notation to denote representations. For instance, $\begin{smallmatrix} S^1 & S^2 & U & W \\ T^1 & T^2 & F & 1 \end{smallmatrix}$ denotes the representation

$$(\mathcal{K}^1 \otimes \mathcal{K}^2 \otimes \mathcal{L} \otimes \mathcal{H}, \mathbb{S}^1 \oplus \mathbb{S}^2 \oplus \mathbb{U} \oplus \mathbb{W}, \mathbb{T}^1 \oplus \mathbb{T}^2 \oplus \mathbb{F} \oplus 1).$$

All braiding operators in this diagram exist because the Hilbert space \mathcal{L} is on the top strand. They are intertwiners of braided representations, compare (3.7). The remaining arrows are also intertwiners of braided representations by the proof of Lemma 3.2. Before we show that the diagram in Figure 1 commutes, we deduce the condition (2.8.2) from it. The arrow from the (2, 1)-entry to the (2, 5)-entry in Figure 1 along the top boundary is the intertwiner

$$\begin{aligned} \mathbb{T}^1 \oplus \mathbb{T}^2 \oplus 1_{\mathcal{H}}: (\mathcal{K}^1 \otimes \mathcal{K}^2 \otimes \mathcal{H} \otimes \mathcal{L}, \mathbb{S}^1 \oplus \mathbb{S}^2 \oplus \mathbb{W} \oplus \mathbb{U}, \mathbb{T}^1 \oplus \mathbb{T}^2 \oplus 1 \oplus \mathbb{F}) \\ \rightarrow (\mathcal{K}^1 \otimes \mathcal{K}^2 \otimes \mathcal{H} \otimes \mathcal{L}, \mathbb{S}^1 \oplus \mathbb{S}^2 \oplus \mathbb{W} \oplus \mathbb{U}, 1 \oplus 1 \oplus 1 \oplus \mathbb{F}), \end{aligned}$$

compare the proof in Lemma 3.2 that the tensor product is associative. And the arrow going downward from there is $(\mathbb{S}^1 \oplus \mathbb{S}^2)_{123}$. So the composite arrow is the absorbing intertwiner for the tensor product $(\mathcal{K}^1 \otimes \mathcal{K}^2, \mathbb{S}^1 \oplus \mathbb{S}^2, \mathbb{T}^1 \oplus \mathbb{T}^2)$. Similarly, the arrows labeled A^2 and A^1 are the absorbing intertwiners for $(\mathcal{K}^2, \mathbb{S}^2, \mathbb{T}^2)$ and $(\mathcal{K}^1, \mathbb{S}^1, \mathbb{T}^1)$, respectively. Hence the commutativity of the boundary of the diagram in Figure 1 is exactly (2.8.2).

Now we check that the diagram in Figure 1 commutes. The four triangles of braiding operators commute because the braiding operators have enough of the properties of a braided monoidal category, compare the proof in Lemma 3.2. The two pentagons with A^1 and A^2 as one of the faces commute by definition of our

absorbing intertwiners. The two parallelograms with \mathbb{S}^2 and braiding operators commute because the braiding operators are natural with respect to intertwiners of \mathbb{W} -representations. The square with \mathbb{T}_{12}^1 and \mathbb{S}_{34}^2 commutes because we operate on different legs. Finally, we consider the square involving \mathbb{T}^1 and the braiding operator $\kappa^2 \times \mathcal{L}$. Here \mathcal{K}^2 carries the trivial representation of \mathbb{W} , so that the braiding is just the tensor flip Σ_{23} . Thus the square commutes, and now we have seen that the entire diagram commutes. \square

Theorem 3.4. *The operator*

$$\mathbb{W}^C := \mathbb{W}_{13} \mathbb{U}_{23} \hat{\mathbb{V}}_{34}^* \mathbb{F}_{24} \hat{\mathbb{V}}_{34} \in \mathcal{U}(\mathcal{H} \otimes \mathcal{L} \otimes \mathcal{H} \otimes \mathcal{L})$$

is a multiplicative unitary such that there is a fully faithful, strict tensor functor $\Phi: \mathfrak{Rep}(\mathbb{W}, \mathbb{U}, \hat{\mathbb{V}}, \mathbb{F}) \rightarrow \mathfrak{Rep}(\mathbb{W}^C)$ with $\mathfrak{For} \circ \Phi = \mathfrak{For}$. The functor Φ maps a representation $(\mathcal{K}, \mathbb{S}, \mathbb{T})$ of $(\mathbb{W}, \mathbb{U}, \hat{\mathbb{V}}, \mathbb{F})$ to the following representation of \mathbb{W}^C :

$$\mathbb{S}_{12}(\mathbb{T} \oplus 1_{\mathcal{H}}) = \mathbb{S}_{12} \hat{\mathbb{V}}_{23}^* \mathbb{T}_{13} \hat{\mathbb{V}}_{23} \in \mathcal{U}(\mathcal{K} \otimes \mathcal{H} \otimes \mathcal{L}).$$

The functor Φ is an isomorphism of categories if \mathbb{W}^C and \mathbb{W} are manageable.

The manageability of \mathbb{W}^C is expressed in [5] in terms of the braided multiplicative unitary $(\mathbb{W}, \mathbb{U}, \hat{\mathbb{V}}, \mathbb{F})$.

Proof. We have found a natural right absorber (ρ, A) in Proposition 3.3. Proposition 2.11 shows that A^ρ is a multiplicative unitary and that $x \mapsto A^x$ is a fully faithful, strict tensor functor $\mathfrak{Rep}(\mathbb{W}, \mathbb{U}, \hat{\mathbb{V}}, \mathbb{F}) \rightarrow \mathfrak{Rep}(A^\rho)$. By definition, $A^x = \mathbb{S}_{12}(\mathbb{T} \oplus 1_{\mathcal{H}}) = \mathbb{S}_{12}(\mathcal{L} \times \mathcal{H})_{23} \mathbb{T}_{12}(\mathcal{H} \times \mathcal{L})_{23}$ and, in particular,

$$A^\rho = (\mathbb{W} \oplus \mathbb{U})_{123} (1_{\mathcal{H}} \oplus \mathbb{F} \oplus 1_{\mathcal{H}}) = \mathbb{W}_{13} \mathbb{U}_{23} (\mathcal{L} \times \mathcal{H})_{34} \mathbb{F}_{23} (\mathcal{H} \times \mathcal{L})_{34}.$$

The definition and (3.4) give The braiding unitary $\mathcal{L} \times \mathcal{H} \in \mathcal{U}(\mathcal{L} \otimes \mathcal{H}, \mathcal{H} \otimes \mathcal{L})$ is equal to $Z\Sigma$ for the unique unitary $Z \in \mathcal{U}(\mathcal{H} \otimes \mathcal{L})$ that verifies the following equation:

$$Z_{13} = \hat{\mathbb{V}}_{23} \mathbb{W}_{12}^* \hat{\mathbb{V}}_{23}^* \mathbb{W}_{12} \quad \text{in } \mathcal{U}(\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{L}),$$

compare [4, (6.10)]. As a left representation of \mathbb{W} , $\hat{\mathbb{V}}$ satisfies $\hat{\mathbb{V}}_{23} \mathbb{W}_{12} \hat{\mathbb{V}}_{23}^* = \mathbb{W}_{12} \hat{\mathbb{V}}_{13}$. Hence $Z_{13} = \hat{\mathbb{V}}_{13}^*$. Since $\Sigma_{34} \mathbb{F}_{23} \Sigma_{34} = \mathbb{F}_{24}$, we get the asserted formulas for A^x and A^ρ . We still have to prove that every representation of \mathbb{W}^C comes from one of $(\mathbb{W}, \mathbb{U}, \hat{\mathbb{V}}, \mathbb{F})$. This will take a while and require some further results. This proof will be completed at the end of this article. \square

Proposition 3.5. *The operators $\mathbb{W}_{13} \mathbb{U}_{23} \in \mathcal{U}(\mathcal{H} \otimes \mathcal{L} \otimes \mathcal{H})$ and $\mathbb{W}_{12} \in \mathcal{U}(\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{L})$ are bicharacters from the multiplicative unitary \mathbb{W}^C to $\mathbb{W} \in \mathcal{U}(\mathcal{H} \otimes \mathcal{H})$ and back, whose composite from \mathbb{W}^C to itself is equal to $\mathbb{P} := \mathbb{W}_{13} \mathbb{U}_{23} \in \mathcal{U}(\mathcal{H} \otimes \mathcal{L} \otimes \mathcal{H} \otimes \mathcal{L})$. Equivalently, the following pentagon-like equations hold:*

$$\mathbb{P}_{23} \mathbb{W}_{12}^C = \mathbb{W}_{12}^C \mathbb{P}_{13} \mathbb{P}_{23}, \quad \mathbb{W}_{23}^C \mathbb{P}_{12} = \mathbb{P}_{12} \mathbb{P}_{13} \mathbb{W}_{23}^C, \quad \mathbb{P}_{23} \mathbb{P}_{12} = \mathbb{P}_{12} \mathbb{P}_{13} \mathbb{P}_{23}.$$

Proof. There are two obvious strict tensor functors between the Hilbert space tensor categories $\mathfrak{Rep}(\mathbb{W}, \mathbb{U}, \hat{\mathbb{V}}, \mathbb{F})$ and $\mathfrak{Rep}(\mathbb{W})$, namely, the forgetful functor

$$\mathfrak{Rep}(\mathbb{W}, \mathbb{U}, \hat{\mathbb{V}}, \mathbb{F}) \rightarrow \mathfrak{Rep}(\mathbb{W}), \quad (\mathcal{K}, \mathbb{S}, \mathbb{T}) \mapsto (\mathcal{K}, \mathbb{S}),$$

and the functor

$$\mathfrak{Rep}(\mathbb{W}) \rightarrow \mathfrak{Rep}(\mathbb{W}, \mathbb{U}, \hat{\mathbb{V}}, \mathbb{F}), \quad (\mathcal{K}, \mathbb{S}) \mapsto (\mathcal{K}, \mathbb{S}, 1_{\mathcal{K}}).$$

The definitions imply immediately that these are strict tensor functors that are compatible with the forgetful functors to \mathfrak{Hilb} . Both tensor categories involved have natural right absorbers, and the associated multiplicative unitaries are \mathbb{W}^C and \mathbb{W} , respectively. Proposition 2.15 produces bicharacters from strict tensor functors like the ones above. Furthermore, the composite functor on $\mathfrak{Rep}(\mathbb{W})$ is the identity.

Correspondingly, the composite bicharacter from \mathbb{W} to itself is the bicharacter that describes the identity functor, which is \mathbb{W} itself. And the composite bicharacter from \mathbb{W}^C to itself is idempotent, which means that it satisfies the pentagon equation itself. It remains to make explicit which bicharacters we get from the formulas in Proposition 2.15.

The bicharacter describing the functor $\mathfrak{Rep}(\mathbb{W}, \mathbb{U}, \hat{\mathbb{V}}, \mathbb{F}) \rightarrow \mathfrak{Rep}(\mathbb{W})$ is the canonical unitary intertwiner

$$\mathbb{W} \oplus \mathbb{U} = \mathbb{W}_{13} \mathbb{U}_{23} : (\mathcal{H} \otimes \mathcal{L}, \mathbb{W} \oplus \mathbb{U}) \otimes (\mathcal{H}, \mathbb{W}) \rightarrow (\mathcal{H} \otimes \mathcal{L}, 1) \otimes (\mathcal{H}, \mathbb{W}),$$

that is, we get $\mathbb{W}_{13} \mathbb{U}_{23} \in \mathcal{U}(\mathcal{H} \otimes \mathcal{L} \otimes \mathcal{H})$.

The bicharacter describing the functor $\mathfrak{Rep}(\mathbb{W}) \rightarrow \mathfrak{Rep}(\mathbb{W}, \mathbb{U}, \hat{\mathbb{V}}, \mathbb{F})$ is the natural isomorphism

$$(\mathcal{H}, \mathbb{W}, 1) \otimes (\mathcal{H} \otimes \mathcal{L}, \mathbb{W} \otimes \mathbb{U}, 1 \oplus \mathbb{F}) \rightarrow (\mathcal{H}, 1, 1) \otimes (\mathcal{H} \otimes \mathcal{L}, \mathbb{W} \otimes \mathbb{U}, 1 \oplus \mathbb{F})$$

described during the proof of Proposition 3.3. Since the representation of \mathbb{F} is 1 here, this simplifies to the unitary $\mathbb{W}_{12} \in \mathcal{U}(\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{L})$.

By the definition of the composition of bicharacters in [2, Definition 3.5], the composite bicharacter from \mathbb{W}^C to itself is $\mathbb{W}_{13} \mathbb{U}_{23}$ if and only if the following equation holds:

$$\mathbb{W}_{34}(\mathbb{W}_{13} \mathbb{U}_{23}) = (\mathbb{W}_{13} \mathbb{U}_{23})(\mathbb{W}_{14} \mathbb{U}_{24}) \mathbb{W}_{34} \in \mathcal{U}(\mathcal{H} \otimes \mathcal{L} \otimes \mathcal{H} \otimes \mathcal{H} \otimes \mathcal{L}).$$

Indeed, the representation property of \mathbb{U} and the pentagon equation for \mathbb{W} give

$$\mathbb{W}_{13} \mathbb{U}_{23} \mathbb{W}_{14} \mathbb{U}_{24} \mathbb{W}_{34} = \mathbb{W}_{13} \mathbb{W}_{14} \mathbb{U}_{23} \mathbb{U}_{24} \mathbb{W}_{34} = \mathbb{W}_{13} \mathbb{W}_{14} \mathbb{W}_{34} \mathbb{U}_{23} = \mathbb{W}_{34} \mathbb{W}_{13} \mathbb{U}_{23}$$

as desired. The general theory says that all these unitaries are bicharacters and that the bicharacter \mathbb{P} is idempotent, that is, satisfies the pentagon equation. \square

It remains to prove that every representation of \mathbb{W}^C comes from a representation of the braided multiplicative unitary if \mathbb{W}^C is manageable. That is, we want it to be of the form $\mathbb{S}_{12} \hat{\mathbb{V}}_{23}^* \mathbb{T}_{13} \hat{\mathbb{V}}_{23}$ for some representation $(\mathcal{K}, \mathbb{S}, \mathbb{T})$ of $(\mathbb{W}, \mathbb{U}, \hat{\mathbb{V}}, \mathbb{F})$. So we start with a representation $(\mathcal{K}, \mathbb{A})$ of \mathbb{W}^C . The Hilbert space must remain \mathcal{K} . We have described the functor

$$\mathfrak{Rep}(\mathbb{W}, \mathbb{U}, \hat{\mathbb{V}}, \mathbb{F}) \rightarrow \mathfrak{Rep}(\mathbb{W}), \quad (\mathcal{K}, \mathbb{S}, \mathbb{T}) \mapsto (\mathcal{K}, \mathbb{S}),$$

through the bicharacter $\mathbb{W}_{13} \mathbb{U}_{23}$ from \mathbb{W}^C to \mathbb{W} in Proposition 3.5. Proposition 2.15 shows that $\mathbb{S} \in \mathcal{U}(\mathcal{K} \otimes \mathcal{H})$ satisfies

$$(3.8) \quad (\mathbb{W}_{24} \mathbb{U}_{34}) \mathbb{A}_{123} = \mathbb{A}_{123} \mathbb{S}_{14} (\mathbb{W}_{24} \mathbb{U}_{34}) \in \mathcal{U}(\mathcal{K} \otimes \mathcal{H} \otimes \mathcal{L} \otimes \mathcal{H}).$$

This equation determines \mathbb{S} uniquely, and it exists because the multiplicative unitary \mathbb{W} is manageable. Now \mathbb{T} should satisfy $\mathbb{A}_{123} = \mathbb{S}_{12} \hat{\mathbb{V}}_{23}^* \mathbb{T}_{13} \hat{\mathbb{V}}_{23}$, that is,

$$\mathbb{T}_{13} = \hat{\mathbb{V}}_{23} \mathbb{S}_{12}^* \mathbb{A}_{123} \hat{\mathbb{V}}_{23}^* \quad \text{in } \mathcal{U}(\mathcal{K} \otimes \mathcal{H} \otimes \mathcal{L}).$$

It remains to prove, first, that the right hand side has trivial second leg, so that it comes from a unitary $\mathbb{T} \in \mathcal{U}(\mathcal{K} \otimes \mathcal{L})$; and, secondly, that $(\mathcal{K}, \mathbb{S}, \mathbb{T})$ is a representation of $(\mathbb{W}, \mathbb{U}, \hat{\mathbb{V}}, \mathbb{F})$. Since these computations are quite unpleasant, we proceed indirectly. For the purpose of this proof, we say that a representation of \mathbb{W}^C comes from a braided representation if it belongs to the image of the functor $\mathfrak{Rep}(\mathbb{W}, \mathbb{U}, \hat{\mathbb{V}}, \mathbb{F}) \rightarrow \mathfrak{Rep}(\mathbb{W})$.

Lemma 3.6. *Let $(\mathcal{K}^1, \mathbb{A}^1)$ and $(\mathcal{K}^2, \mathbb{A}^2)$ be representations of \mathbb{W}^C . If $(\mathcal{K}^1, \mathbb{A}^1)$ and $(\mathcal{K}^1 \otimes \mathcal{K}^1, \mathbb{A}^1 \oplus \mathbb{A}^2)$ come from braided representations, then so does $(\mathcal{K}^2, \mathbb{A}^2)$.*

Proof. We define \mathbb{S}^i and \mathbb{T}^i for $i = 1, 2$ as above. We know that $(\mathcal{K}^1, \mathbb{S}^1, \mathbb{T}^1)$ is a braided representation. But at first, we only know $\mathbb{T}^2 \in \mathcal{U}(\mathcal{K}^2 \otimes \mathcal{L} \otimes \mathcal{H})$. We may, nevertheless, recycle the diagram in Figure 1, treating it as a diagram in $\mathfrak{Rep}(\mathbb{W})$ only, and replacing the top left arrow \mathbb{T}_{23}^2 by \mathbb{T}_{234}^2 . The two pentagons still commute by definition of $\mathbb{S}^i, \mathbb{T}^i$. The four triangles of braiding operators in Figure 1 commute as before. So do the parallelograms containing \mathbb{S}_{23}^2 and braiding operators, and the two squares in the middle: this only needs $(\mathbb{S}^1, \mathbb{T}^1)$ to be a braided representation, which we have assumed. Hence the entire diagram commutes. The composite arrow from the $(2, 1)$ -entry to the $(5, 4)$ -entry is the tensor product representation $\mathbb{A}^1 \oplus \mathbb{A}^2$. We have assumed that this comes from a braided representation. This must be of the form $(\mathbb{S}^1 \oplus \mathbb{S}^2, \mathbb{T})$ for some $\mathbb{T} \in \mathcal{U}(\mathcal{K}^1 \otimes \mathcal{K}^2 \otimes \mathcal{L})$. Hence

$$(\mathcal{K}^2 \times \mathcal{L})_{23} \mathbb{T}_{12}^1 (\mathcal{L} \times \mathcal{K}^2)_{23} \mathbb{T}_{234}^2 = \mathbb{T}_{123}.$$

Therefore, \mathbb{T}_{234}^2 acts trivially on the fourth leg. So $\mathbb{A}^2 = \mathbb{S}_{12}^2 \hat{\mathbb{V}}_{23}^* \mathbb{T}_{13}^2 \hat{\mathbb{V}}_{23}$ for some $\mathbb{T}^2 \in \mathcal{U}(\mathcal{K}^2 \otimes \mathcal{L})$. In the proof of Lemma 3.2, we have shown that a unitary \mathbb{T} in $\mathcal{U}(\mathcal{K} \otimes \mathcal{L})$ together with a representation $(\mathcal{K}, \mathbb{S})$ of \mathbb{W} is a braided representation if and only if \mathbb{T} is an intertwiner from $\mathbb{T} \oplus \mathbb{F}$ to $\mathbb{T} \oplus 1_{\mathcal{L}}$. Therefore, $\mathbb{T}^1 \oplus \mathbb{T}^2 = (\mathcal{L} \times \mathcal{K}^2)_{23} \mathbb{T}_{12}^1 (\mathcal{K}^2 \times \mathcal{L})_{23} \mathbb{T}_{23}^2$ and \mathbb{T}^1 are intertwiners of braided representations. So are the braiding operators, compare (3.7). Hence $\mathbb{T}_{23}^2 \in \mathcal{U}(\mathcal{K}^1 \otimes \mathcal{K}^2 \otimes \mathcal{L})$ is an intertwiner of braided representations. Then so is \mathbb{T}^2 itself. This means that $(\mathbb{S}^2, \mathbb{T}^2)$ is a braided representation. \square

Since \mathbb{W}^C is manageable, Proposition 2.17 shows that $\mathfrak{Rep}(\mathbb{W})$ contains a (natural) left absorber \mathbb{A}^1 . Even more, the proof shows that we may choose \mathbb{A}^1 to be isomorphic to a direct sum of copies of \mathbb{W}^C . By definition, \mathbb{W}^C comes from the braided representation $(\mathcal{H} \otimes \mathcal{L}, \mathbb{W} \oplus \mathbb{U}, 1_{\mathcal{H}} \oplus \mathbb{F})$. Hence the direct sum of countably many summands of \mathbb{W}^C also comes from a braided representation. Since $\mathbb{A}^1 \oplus \mathbb{A}^2 \cong \mathbb{A}^1 \oplus 1_{\mathcal{K}^2} \cong \mathbb{A}^1$ for any representation $(\mathcal{K}^2, \mathbb{A}^2)$, $\mathbb{A}^1 \oplus \mathbb{A}^2$ also comes from a braided representation. Now Lemma 3.6 shows that any representation $(\mathcal{K}^2, \mathbb{A}^2)$ of \mathbb{W}^C comes from a braided representation. This finishes the proof of Theorem 3.4.

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